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SPECIFYING NODES AT MULTIPLE LOCATIONS FOR ANY NORMAL MODE OF A LINEAR ELASTIC STRUCTURE

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Spring–mass oscillators have long been used as vibration absorbers to quench excess vibration in structural systems. In this paper, sprung masses are used as a passive means of inducing multiple nodes for any normal mode of an arbitrarily supported, linear elastic structure. When the parameters of the elastically mounted masses are properly chosen, their attachment locations can be made to coincide exactly with the nodes of the structure, thereby allowing nodes to be imposed at multiple locations anywhere along the combined assembly. Moreover, when the desired node locations are closely spaced, it is possible to specify a region of nearly zero amplitudes for a particular normal mode. A procedure to guide the proper selection of the spring–mass parameters in order to induce multiple nodes is outlined in detail, and numerical experiments are performed to verify the utility of the proposed scheme of imposing nodes at multiple locations.

1. INTRODUCTION

The vibration of a linear elastic structure carrying any number of sprung masses has received considerable interest in recent years, and hence only a few selected references are given here. Using the Lagrange multiplier approach, Dowell [1] obtained the frequency equation of a beam with an elastically mounted mass. Using the dynamic Green function approach, Nicholson and Bergman [2] performed a frequency analysis on two combined systems, one consisting of a fixed–free beam connected at discrete points to oscillators with no rigid body degree of freedom, and one consisting of a simply supported beam connected at discrete points to oscillators with a rigid body degree of freedom. Using the assumed-modes method, Ercoli and Laura [3] studied the effects of concentrated masses elastically mounted to a beam on the frequencies of the system. Using the Green functions method, Kukla and Posiadala [4] obtained closed-form expressions for the frequency equations of beams with sprung masses. In reference [5], Gürgöze formulated two schemes to compute the natural frequencies of a Euler–Bernoulli beam to which several spring–mass systems are attached in span. He first used the Lagrange multipliers formalism to obtain the characteristic determinant of the system, with which he used to compute the natural frequencies of the combined structure. He then used the assumed-modes method, in addition to a co-ordinate transformation, to obtain yet another characteristic determinant that can be used to solve for the natural frequencies. Wu and Chou [6] introduced a numerical technique to obtain the exact solutions for the lower modes of vibration of a uniform beam with various boundary conditions carrying any number of sprung masses. They first formulated the coefficient matrix for a beam element, then used the conventional finite element assembly technique to determine the overall coefficient matrix, from which they solved for the natural frequencies and mode shapes of the entire structure.

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In all of the previously mentioned work, the authors employed various approaches to compute the natural frequencies of beams carrying elastically mounted masses. However, none considered utilizing multiple spring–mass systems to induce multiple nodes for any normal mode of a linear elastic structure. In reference [7], Cha and Pierre used a chain of oscillators as a means to passively impose a node for the normal modes of any arbitrarily supported, linear elastic structure. The desired node can either coincide with the oscillator chain or it can be located elsewhere. When the oscillator chain and the node are collocated, a node can be induced at any location along the linear structure for multiple normal modes. However, when they are not collocated, then a node can only be imposed for certain normal modes. A procedure to guide the proper selection of the oscillator chain parameters for the purpose of inducing a node for multiple normal modes was outlined in detail.

In a recent technical note [8], the present author proposed an approach to compute the natural frequencies of an arbitrarily supported, linear elastic structure to which any number of elastically mounted masses are attached. Compared with other commonly used methods, the proposed scheme offered several distinct advantages. Specifically, the proposed approach was simple to code, can include any number of sprung masses, and can be extended to accommodate any linear elastic structure with any arbitrary boundary conditions. In addition to the benefits cited previously, it will be shown that the proposed formalism also allows multiple nodes to be induced anywhere along the structure for any normal mode. While this inverse problem of imposing multiple nodes is not amenable to other solution schemes, it can be easily solved using the approach outlined in reference [8].

In this paper, a series of sprung masses will be used to induce multiple nodes for any normal mode of an arbitrarily supported, linear elastic structure. By selecting the appropriate sprung masses, their attachment locations can be made to coincide exactly with the nodes of the structure, thereby allowing the locations of the nodes to be specified anywhere along the structure and for any normal mode. This is beneficial because it would allow sensitive instruments to be placed near or at nodes where there are little or no vibration. A procedure will be proposed to guide the selection of the spring–mass parameters in order to induce multiple nodes at any desired locations along the structure. Numerical experiments will be performed to illustrate how the proposed scheme can be used to impose any number of nodes, and to show the utility of the proposed scheme in passively suppressing excess vibration about some small region along the structure for any specified normal mode.

2. THEORY

Consider the free vibration of an arbitrarily supported, linear structure to which S-sprung masses are attached as shown in Figure 1. Using the assumed-modes method, the physical

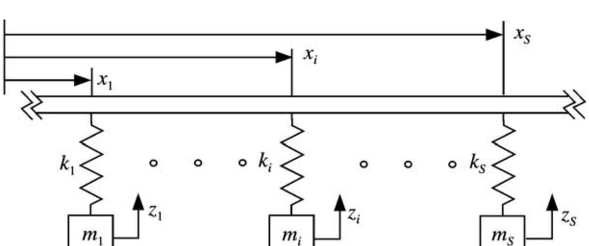

Figure 1. An arbitrarily supported, linear elastic structure carrying any number of sprung masses.

deflection of the structure at any point x along the structure is given by

$$w(x, t) = \sum_{i=1}^{N} \phi_i(x)\eta_i(t), \tag{1}$$

where the $\phi_i(x)$ are the eigenfunctions of the linear structure (the elastica without any sprung masses) that serve as the basis functions for this approximate solution, the $\eta_i(t)$ are the corresponding generalized co-ordinates, and N is the number of modes used in the assumed-modes expansion. The total kinetic and potential energies of the combined system, defined as the linear elastic structure carrying the elastically mounted masses, are given by

$$T = \frac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2(t) + \frac{1}{2}\sum_{i=1}^{S} m_i\dot{z}_t^2(t) \tag{2}$$

and

$$V = \frac{1}{2}\sum_{i=1}^{N} K_i\eta_i^2(t) + \frac{1}{2}\sum_{i=1}^{S} k_i[z_i(t) - w(x_i, t)]^2, \tag{3}$$

where M_i and K_i are, respectively, the generalized masses and stiffnesses of the linear elastica, m_i and k_i are, respectively, the mass and spring stiffness of the ith oscillator, $z_i(t)$ is its displacement, S is the total number of sprung masses attached to the elastica, an overdot denotes a derivative with respect to time, and $w(x_i, t)$ represents the lateral displacement of the beam at x_i.

Applying Lagrange's equations and assuming simple harmonic motion,

$$\eta_i(t) = \bar{\eta}_i e^{j\omega t}, \qquad z_i(t) = \bar{z}_i e^{j\omega t}, \tag{4}$$

where $j = \sqrt{-1}$ and ω is the natural frequency, the modes of vibration for the system of Figure 1 correspond to the solutions of the following generalized eigenvalue problem

$$\begin{bmatrix} [\mathscr{K}] & [R] \\ [R]^T & [k] \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathbf{z}} \end{bmatrix} = \omega^2 \begin{bmatrix} [\mathscr{M}] & [0] \\ [0]^T & [m] \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{\eta}} \\ \bar{\mathbf{z}} \end{bmatrix}, \tag{5}$$

where $\bar{\boldsymbol{\eta}} = [\bar{\eta}_1\ \bar{\eta}_2 \cdots \bar{\eta}_N]^T$ is the vector of generalized co-ordinates, $\bar{\mathbf{z}} = [\bar{z}_1\ \bar{z}_2 \cdots \bar{z}_S]^T$ is the vector of mass displacements, and the $S \times S$ matrices $[m]$ and $[k]$ are both diagonal, whose ith elements are given by m_i and k_i respectively. The $N \times N [\mathscr{M}]$ and $[\mathscr{K}]$ matrices of equation (5) are

$$[\mathscr{M}] = [M^d], \qquad [\mathscr{K}] = [K^d] + \sum_{i=1}^{S} k_i \boldsymbol{\phi}(x_i)\boldsymbol{\phi}^T(x_i), \tag{6}$$

where $[M^d]$ and $[K^d]$ are diagonal matrices whose ith elements are M_i and K_i, vector $\boldsymbol{\phi}(x_i)$ consists of the eigenfunctions of the linear elastic structure evaluated at x_i,

$$\boldsymbol{\phi}(x_i) = [\phi_1(x_i) \quad \phi_2(x_i) \cdots \phi_N(x_i)]^T \tag{7}$$

and the $N \times M$ matrix $[R]$ is given by

$$[R] = [-k_1\boldsymbol{\phi}(x_1) \cdots - k_i\boldsymbol{\phi}(x_i) \cdots - k_S\boldsymbol{\phi}(x_S)]. \tag{8}$$

Note that $[\mathscr{M}]$ is a diagonal matrix and $[\mathscr{K}]$ is a diagonal matrix modified by S rank one matrices. Once the linear elastic structure and the attachment locations are specified, and the spring–mass parameters are given, the natural frequencies of the combined system can be readily obtained by solving the $(N+S)\times(N+S)$ generalized eigenvalue problem of equation (5).

The characteristic determinant associated with equation (5) can be reduced by simple algebraic manipulation. Using equation (5), the $\bar{z}_i$ are found to be

$$\bar{z}_i = \frac{k_i \boldsymbol{\phi}^{\mathrm{T}}(x_i)}{k_i - \omega^2 m_i}\,\bar{\boldsymbol{\eta}}, \quad i = 1, 2, \ldots, S. \tag{9}$$

Substituting the expressions of equation (9) into equation (5), the following generalized eigenvalue problem, of size $N \times N$, is obtained:

$$\left\{[K^d] + \sum_{i=1}^{S} \sigma_i \boldsymbol{\phi}(x_i)\boldsymbol{\phi}^{\mathrm{T}}(x_i)\right\}\bar{\boldsymbol{\eta}} = \omega^2 [M^d]\bar{\boldsymbol{\eta}}, \tag{10}$$

where

$$\sigma_i = \frac{k_i m_i \omega^2}{\omega^2 m_i - k_i}. \tag{11}$$

Expanding equation (10), the natural frequencies of the system are given by the solution of the following characteristic determinant

$$\det\left\{[K^d] - \omega^2 [M^d] + \sum_{i=1}^{S} \sigma_i \boldsymbol{\phi}(x_i)\boldsymbol{\phi}^{\mathrm{T}}(x_i)\right\} = 0, \tag{12}$$

which can be shown [9] to be identical to

$$\det\{[K^d] - \omega^2 [M^d]\}\det[B] = \left\{\prod_{i=1}^{N} (K_i - \omega^2 M_i)\right]\det[B] = 0, \tag{13}$$

where the (i,j)th element of $[B]$, of size $S \times S$, is given by

$$b_{ij} = \sum_{r=1}^{N} \frac{\phi_r(x_i)\phi_r(x_j)}{K_r - \omega^2 M_r} + \frac{1}{\sigma_i}\delta_i^j, \quad i, j = 1, 2, \ldots, S, \tag{14}$$

and δ_i^j represents the Kronecker delta. If $\omega^2 \neq K_i/M_i$, which occurs when the attachment locations for the sprung masses are distinct from the nodes of any component mode, $\phi_i(x)$, then equation (13) reduces to

$$\det[B] = 0, \tag{15}$$

the same result as equation (25) that Gürgöze [5] obtained by using the Lagrange multipliers formalism.

Suppose for a given application, it is desired to impose multiple nodes along the linear elastic structure, at $x_1, x_2, \ldots, x_S$, for a given normal mode. This can be accomplished by attaching appropriately chosen spring–mass systems to the elastica at the specified locations. If the attachment points of the sprung masses do not coincide with the nodes of the elastica, then the natural frequencies of the combined system, which consists of the linear elastic structure and the attached oscillators, are given by the roots of the characteristic determinant of equation (15). From equation (5), the following expressions are obtained that relate the displacements of the sprung masses and their attachment locations:

$$z_i(k_i - \omega^2 m_i) = k_i \boldsymbol{\phi}_i^{\mathrm{T}} \boldsymbol{\eta} = k_i \sum_{r=1}^{N} \phi_r(x_i)\eta_r(t) = k_i w(x_i, t). \quad (16)$$

Note that if

$$k_i = \omega^2 m_i, \quad i = 1, \ldots, S, \quad (17)$$

which implies that when a natural frequency of the combined system coincides with the natural frequencies of the grounded oscillators, then

$$w(x_i, t) = 0, \quad i = 1, \ldots, S, \quad (18)$$

indicating that the attachment locations, x_i, are all nodes. To have all the x_i as nodes simultaneously for the same normal mode, equation (17) has to be satisfied for all i's, in which case equation (15) simplifies to

$$\det[B'] = 0, \quad (19)$$

where the (i, j)th element of $[B']$, of size $S \times S$, is given by

$$b'_{ij} = \sum_{r=1}^{N} \frac{\phi_r(x_i)\phi_r(x_j)}{K_r - \omega_{osc}^2 M_r}, \quad i, j = 1, 2, \ldots, S, \quad (20)$$

and

$$\omega_{osc}^2 = \frac{k_i}{m_i}, \quad i = 1, \ldots, S. \quad (21)$$

Based on the above discussion, the following procedure to select the required oscillator parameters in order to induce nodes at x_i for any given normal mode can be formulated:

1. For given node locations, x_i, solve equation (20) for all the possible $(\omega_{osc})_j$'s.
2. These $(\omega_{osc})_j$'s correspond to all the spring–mass combinations that result in nodes at the x_i.
3. Each $(\omega_{osc})_j$ corresponds to a possible natural frequency of the combined system. If the natural frequencies of the combined structure are arranged in the order of increasing magnitude, then the position of $(\omega_{osc})_j$ in the vector of natural frequencies of the combined system dictates the mode at which the x_i become nodes. For instance, if a particular value of $(\omega_{osc})_j$ corresponds to the rth natural frequency of the combined assembly, then by attaching spring–mass systems that satisfy equation (21), nodes at the desired x_i will be induced for the rth normal mode of the combined structure.
4. Depending on which $(\omega_{osc})_j$ value that is selected, nodes at x_i can be imposed for any specific normal mode.

It should be noted that the selection of the spring–mass parameters is not unique. The actual choice is often dictated by the vibration amplitudes of the m_i. Finally, if equation (20) does not yield a value of $(\omega_{osc})_j$ for the given x_i, then it is not possible to induce nodes at the desired x_i simultaneously for any normal mode.

3. RESULTS

The procedures outlined in section 2 will be applied to impose multiple nodes for any normal mode of an arbitrarily supported, linear elastic structure. Without any loss of generality, a simply supported and a fixed–free uniform Euler–Bernoulli beam will be considered. In all of the following numerical examples, $N = 15$ (the number of component modes used in the assumed-modes expansion). In addition, a double precision version of the CMLIB [10] routine *zeroin* was used to find the roots of the characteristic determinant of equation (19), and a double precision version of *rsg* in CMLIB was used to compute the eigensolutions of the combined system by solving equation (5) for any chosen k_i and m_i. Once the eigenvectors are known, the mode shapes can be readily determined.

When the beam is simply supported, let $\phi_i(x)$ of equation (1) be the normalized (with respect to the mass per unit length, ρ, of the beam) eigenfunctions of a uniform simply supported Euler–Bernoulli beam given by

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}} \sin \frac{i\pi x}{L}, \tag{22}$$

such that the generalized masses and stiffnesses of the beam become

$$M_i = 1 \quad \text{and} \quad K_i = (i\pi)^4 EI/(\rho L^4), \tag{23}$$

where E is Young's modulus, I is the moment of inertia of the cross-section of the beam. When the beam is fixed–free, let $\phi_i(x)$ of equation (1) be the normalized eigenfunctions of a uniform fixed–free beam given by

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}} \left(\cos \beta_i x - \cosh \beta_i x + \frac{\sin \beta_i L - \sinh \beta_i L}{\cos \beta_i L + \cosh \beta_i L} (\sin \beta_i x - \sinh \beta_i x) \right) \tag{24}$$

such that the generalized masses and stiffnesses of the beam are

$$M_i = 1 \quad \text{and} \quad K_i = (\beta_i L)^4 EI/(\rho L^4), \tag{25}$$

where $\beta_i L$ satisfies the following transcendental equation:

$$\cos \beta_i L \cosh \beta_i L = -1. \tag{26}$$

Table 1 shows the node locations of the second to the fifth normal modes of a simply supported and a fixed–free uniform Euler–Bernoulli beam.

To illustrate the proposed approach of imposing multiple nodes, consider first a uniform simply supported Euler–Bernoulli beam of length L. For a given application, it is desired that nodes be imposed at $x_1 = 0{\cdot}45L$ and $x_2 = 0{\cdot}88L$ simultaneously for the jth normal mode, where $j \geqslant 3$ because for a simply supported beam, a minimum of two nodes exist only for the third or higher normal modes. Note that the specified node locations are distinct from the nodes of the first five normal modes of the original simply supported beam. To induce nodes at x_1 and x_2, two sprung masses are attached to the beam at x_1 and x_2. The spring–mass parameters are selected such that its grounded natural frequency satisfies equation (19). Table 2 shows the required $(\omega_{osc})_j$ in order to induce nodes at $x_1 = 0{\cdot}45L$ and $x_2 = 0{\cdot}88L$.

Table 2 also shows the first five natural frequencies, the ω_j, of the combined system, consisting of a simply supported Euler–Bernoulli beam carrying two sprung masses, for

TABLE 1

The jth node location, x^j_{node}, for the second to the fifth normal modes of a simply supported and a fixed–free uniform Euler–Bernoulli beam

	Mode number	x^1_{node}	x^2_{node}	x^3_{node}	x^4_{node}
Simply supported	2	0·5000L			
	3	0·3333L	0·6667L		
	4	0·2500L	0·5000L	0·7500L	
	5	0·2000L	0·4000L	0·6000L	0·8000L
Fixed–free	2	0·7834L			
	3	0·5035L	0·8677L		
	4	0·3583L	0·6441L	0·9056L	
	5	0·2788L	0·4999L	0·7232L	0·9265L

TABLE 2

The required oscillator natural frequency, $(\omega_{osc})_j$, in order to induce nodes at $x_1 = 0{\cdot}45L$ and $x_2 = 0{\cdot}88L$ simultaneously

	ω_1	ω_2	ω_3	ω_4	ω_5
$(\omega_{osc})_1 = 56{\cdot}3538$	5·4699	23·1605	<u>56·3538</u>	78·2585	146·5652
$(\omega_{osc})_2 = 95{\cdot}6649$	5·4816	24·1510	59·3156	<u>95·6649</u>	197·2379
$(\omega_{osc})_3 = 210{\cdot}6120$	5·4866	24·5841	60·6981	105·2700	<u>210·6120</u>

Note: The linear elastic structure consists of a simply supported beam. The oscillator parameters are chosen such that $m_1 = m_2 = \rho L$, and $k_1/m_1 = k_2/m_2 = (\omega_{osc})_j^2$. The first five natural frequencies, ω_i, of the combined system for each $(\omega_{osc})_j$ are also given. The natural frequencies are non-dimensionalized by dividing by $\sqrt{EI/\rho L^4}$. Note that each $(\omega_{osc})_j$ corresponds to a natural frequency (see the underlined value) of the combined system.

each $(\omega_{osc})_j$. Because any spring–mass combinations can be used as long as $k_1/m_1 = k_2/m_2 = (\omega_{osc})_j^2$, for simplicity, the masses are chosen such that $m_1 = m_2 = \rho L$. Note that when the oscillator parameters satisfy equation (19), then one of the natural frequency of the combined system coincides with $(\omega_{osc})_j$, in which cases x_1 and x_2 become nodes of the combined structure. From Table 2, note that $\omega_{j+2} = (\omega_{osc})_j$, implying that when the oscillator parameters are chosen such that its natural frequency coincides with the jth solution (arranged in the order of increasing magnitude) of equation (19), then the grounded oscillator natural frequency becomes the $(j+2)$th natural frequency of the combined system. Figure 2(a) shows the third, fourth and fifth normal modes of the combined system, whose attached oscillators have natural frequencies of $(\omega_{osc})_1$, $(\omega_{osc})_2$ and $(\omega_{osc})_3$ respectively. As expected, when the grounded oscillator parameters are selected such that its natural frequency coincides with $(\omega_{osc})_j$, nodes will be induced at x_1 and x_2 for the $(j+2)$th normal mode of the combined assembly.

Table 3 shows the natural frequencies of the combined system for a given $(\omega_{osc})_j$ for the parameters of Table 2, where the masses of the attached oscillators are now given by $m_1 = \rho L$ and $m_2 = 0{\cdot}1\rho L$. Note that except for $\omega_{j+2} = (\omega_{osc})_j$, all the natural frequencies of the combined system are distinct from those of Table 2. Figure 2(b) shows the third, fourth and fifth normal modes of the combined system, whose attached oscillators have natural frequencies of $(\omega_{osc})_1$, $(\omega_{osc})_2$ and $(\omega_{osc})_3$ respectively. Note that while the mode shapes are different from those of Figure 2(a), nodes are still induced at x_1 and x_2.

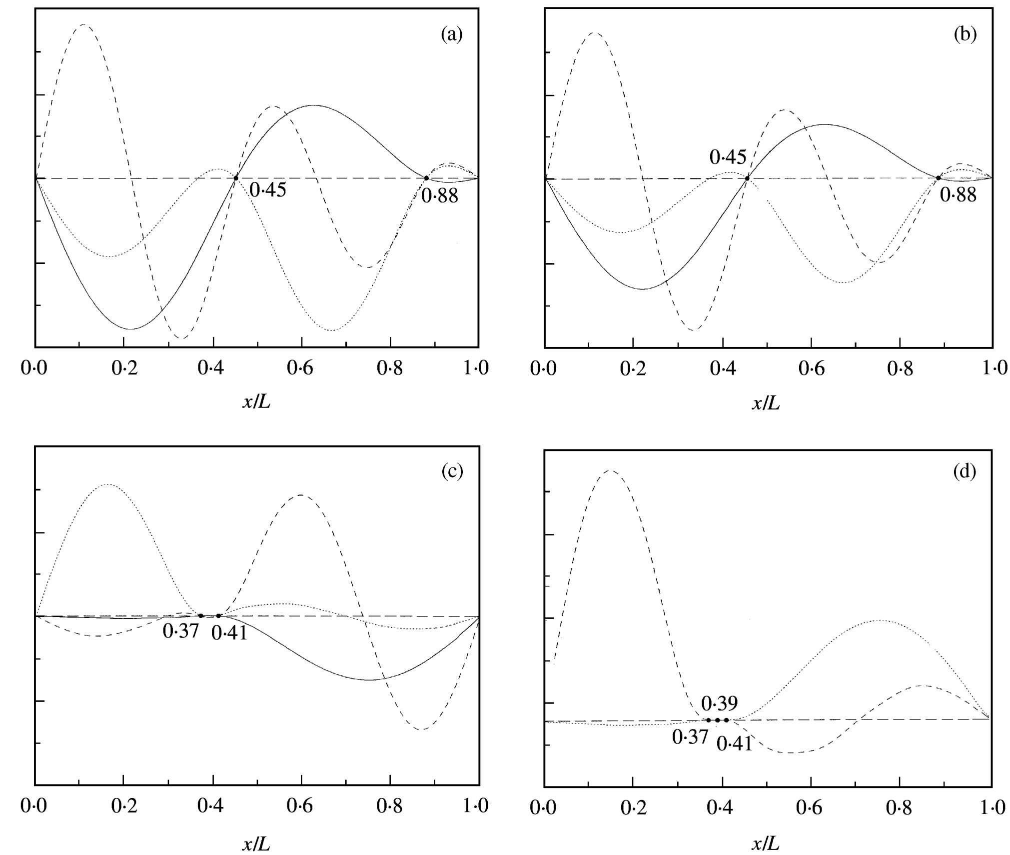

Figure 2. The third to fifth normal modes of a uniform simply supported beam with two sprung masses at $x_1 = 0{\cdot}45L$ and $x_2 = 0{\cdot}88L$. The oscillator parameters are chosen such that (a) $m_1 = m_2 = \rho L$, and $k_1/m_1 = k_2/m_2 = (\omega_{osc})_j^2$; (b) $m_1 = \rho L$, $m_2 = 0{\cdot}1\rho L$, and $k_1/m_1 = k_2/m_2 = (\omega_{osc})_j^2$. (c) The third to fifth normal modes of a uniform simply supported beam with two sprung masses at $x_1 = 0{\cdot}37L$ and $x_2 = 0{\cdot}41L$. The oscillator parameters are chosen such that $m_1 = m_2 = \rho L$, and $k_1/m_1 = k_2/m_2 = (\omega_{osc})_j^2$. For (a)–(c): ——, mode 3; $\cdots\cdot$, mode 4; – – –, mode 5. (d) The fourth and fifth normal modes of a uniform simply supported beam with three sprung masses at $x_1 = 0{\cdot}37L$, $x_2 = 0{\cdot}39L$ and $x_3 = 0{\cdot}41L$. The oscillator parameters are chosen such that $m_1 = m_2 = m_3 = \rho L$, and $k_1/m_1 = k_2/m_2 = k_3/m_3 = (\omega_{osc})_j^2$. For (d): $\cdots\cdot$, mode 4; – – –, mode 5.

TABLE 3

The required oscillator natural frequency, $(\omega_{osc})_j$, in order to induce nodes at $x_1 = 0{\cdot}45L$ and $x_2 = 0{\cdot}88L$ simultaneously

	ω_1	ω_2	ω_3	ω_4	ω_5
$(\omega_{osc})_1 = 56{\cdot}3538$	5·6775	34·4537	<u>56·3538</u>	67·8127	126·5708
$(\omega_{osc})_2 = 95{\cdot}6649$	5·6902	35·5198	65·4353	<u>95.6649</u>	156·2739
$(\omega_{osc})_3 = 210{\cdot}6120$	5·6955	35·8597	68·1325	125·5215	<u>210·6120</u>

Note: The oscillator parameters are chosen such that $m_1 = \rho L$, $m_2 = 0{\cdot}1\rho L$, and $k_1/m_1 = k_2/m_2 = (\omega_{osc})_j^2$.

Table 4 shows the required oscillator natural frequencies, $(\omega_{osc})_j$, in order to have nodes at $x_1 = 0{\cdot}37L$ and $x_2 = 0{\cdot}41L$ of a simply supported beam. Also shown are the first five natural frequencies of the combined system for each $(\omega_{osc})_j$, for $m_1 = m_2 = \rho L$. Note again

TABLE 4

The required oscillator natural frequency, $(\omega_{osc})_j$, in order to induce nodes at $x_1 = 0{\cdot}37L$ and $x_2 = 0{\cdot}41L$ simultaneously

	ω_1	ω_2	ω_3	ω_4	ω_5
$(\omega_{osc})_1 = 42{\cdot}8547$	4·5654	32·1500	<u>42·8547</u>	76·1356	107·1083
$(\omega_{osc})_2 = 107{\cdot}1104$	4·5827	33·2214	73·5027	<u>107·1104</u>	129·1569
$(\omega_{osc})_3 = 140{\cdot}0096$	4·5840	33·2827	75·4466	120·5928	<u>140·0096</u>

Note: The oscillator parameters are chosen such that $m_1 = m_2 = \rho L$, and $k_1/m_1 = k_2/m_2 = (\omega_{osc})_j^2$.

TABLE 5

The required oscillator natural frequency, $(\omega_{osc})_j$, in order to induce nodes at $x_1 = 0{\cdot}37L$, $x_2 = 0{\cdot}39L$ and $x_3 = 0{\cdot}41L$ simultaneously

	ω_1	ω_2	ω_3	ω_4	ω_5
$(\omega_{osc})_1 = 45{\cdot}2341$	3·8690	32·1409	44·8566	<u>45·2341</u>	78·6226
$(\omega_{osc})_2 = 117{\cdot}3629$	3·8793	32·9165	74·0562	112·2079	<u>117·3629</u>

Note: The oscillator parameters are chosen such that $m_1 = m_2 = m_3 = \rho L$, and $k_1/m_1 = k_2/m_2 = k_3/m_3 = (\omega_{osc})_j^2$.

TABLE 6

The required oscillator natural frequency, $(\omega_{osc})_j$, in order to induce nodes at $x_1 = 0{\cdot}93L$ and $x_2 = 0{\cdot}97L$ for simultaneously

	ω_1	ω_2	ω_3	ω_4	ω_5
$(\omega_{osc})_1 = 25{\cdot}2110$	1·2374	<u>16·8127</u>	25·2110	53·6449	96·5203
$(\omega_{osc})_2 = 69{\cdot}5741$	1·2385	17·1635	49·6536	<u>69·5741</u>	118·8332
$(\omega_{osc})_3 = 136{\cdot}5360$	1·2386	17·1942	52·1930	93·1830	<u>136·5360</u>

Note: The linear elastic structure consists of a fixed–free beam. The oscillator parameters are chosen such that $m_1 = m_2 = \rho L$, and $k_1/m_1 = k_2/m_2 = (\omega_{osc})_j^2$. The first five natural frequencies, ω_i, for each $(\omega_{osc})_j$ are also given. The natural frequencies are non-dimensionalized by dividing by $\sqrt{EI/\rho L^4}$.

that $\omega_{j+2} = (\omega_{osc})_j$. Figure 2(c) shows the third to fifth normal modes of the combined system, whose attached oscillators have natural frequencies of $(\omega_{osc})_1$ to $(\omega_{osc})_3$ respectively. Note that when the desired node locations are closely spaced, it is possible to induce a region of very small amplitudes for the third or higher normal modes, depending on which set of spring–mass parameters are selected. This has practical benefits because a region of minimum vibration can be imposed without using any rigid supports.

Table 5 shows the required $(\omega_{osc})_j$ in order to induce nodes at $x_1 = 0{\cdot}37L$, $x_2 = 0{\cdot}39L$ and $x_3 = 0{\cdot}41L$ of a simply supported beam. Also shown are the first five natural frequencies of the combined system as a function of $(\omega_{osc})_j$, for $m_1 = m_2 = m_3 = \rho L$. Note that because three node locations are specified, $\omega_{j+3} = (\omega_{osc})_j$, implying that the j solution, $(\omega_{osc})_j$, induces nodes at the desired locations for the $(j + 3)$th normal mode. Figure 2(d) shows the fourth and fifth normal modes of the combined system, whose attached oscillators have

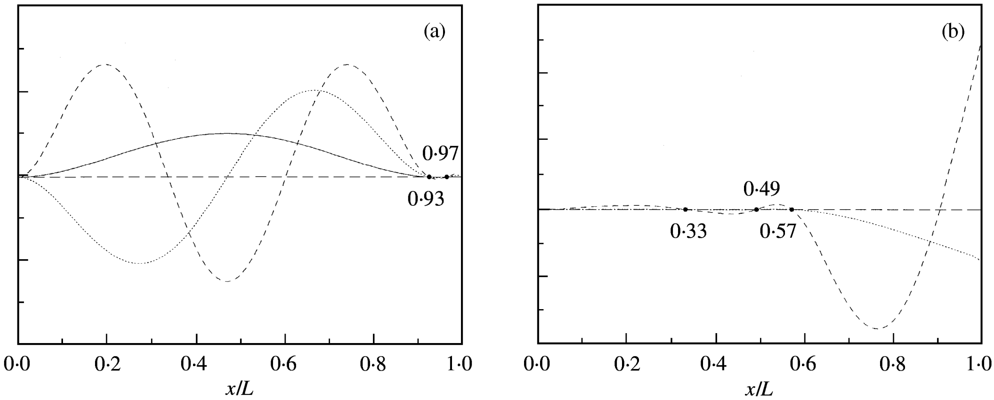

Figure 3. (a) The third to fifth normal modes of a uniform fixed–free beam with two sprung masses at $x_1 = 0{\cdot}93L$ and $x_2 = 0{\cdot}97L$. The oscillator parameters are chosen such that $m_1 = m_2 = \rho L$, and $k_1/m_1 = k_2/m_2 = (\omega_{osc})_j^2$; ——, mode 3; ·····, mode 4; – – –, mode 5. (b) The fourth and fifth normal modes of a uniform fixed–free beam with three sprung masses at $x_1 = 0{\cdot}33L$, $x_2 = 0{\cdot}49L$ and $x_3 = 0{\cdot}57L$. The oscillator parameters are chosen such that $m_1 = m_2 = m_3 = \rho L$, and $k_1/m_1 = k_2/m_2 = k_3/m_3 = (\omega_{osc})_j^2$; ·····, mode 4; – – –, mode 5.

TABLE 7

The required oscillator natural frequency, $(\omega_{osc})_j$, in order to induce nodes at $x_1 = 0{\cdot}33L$, $x_2 = 0{\cdot}49L$ and $x_3 = 0{\cdot}57L$ simultaneously

	ω_1	ω_2	ω_3	ω_4	ω_5
$(\omega_{osc})_1 = 17{\cdot}2147$	2·2947	10·8044	16·2658	17·2147	48·6984
$(\omega_{osc})_2 = 109{\cdot}6164$	2·3065	11·8346	31·1896	71·3566	109·6164

Note: The oscillator parameters are chosen such that $m_1 = m_2 = m_3 = \rho L$, and $k_1/m_1 = k_2/m_2 = k_3/m_3 = (\omega_{osc})_j^2$.

natural frequencies of $(\omega_{osc})_1$ and $(\omega_{osc})_2$ respectively. Compared with the results of Figure 2(c), note that by attaching an additional oscillator at $0{\cdot}39L$, it is possible to induce a region of nearly zero amplitudes for the fourth and fifth normal modes.

Consider now a uniform fixed–free Euler–Bernoulli beam. Table 6 shows the required $(\omega_{osc})_j$ in order to impose nodes at $x_1 = 0{\cdot}93L$ and $x_2 = 0{\cdot}97L$ of a fixed–free beam. The first five natural frequencies of the combined system for each $(\omega_{osc})_j$, are also listed, for $m_1 = m_2 = \rho L$. Because two node locations are imposed, $\omega_{j+2} = (\omega_{osc})_j$. Figure 3(a) shows the third to fifth normal modes of the combined system, whose attached oscillators have natural frequencies of $(\omega_{osc})_1$ to $(\omega_{osc})_3$ respectively. Note that with a properly chosen set of springs and masses, it is possible to induce nearly zero amplitude at $x = L$ for these normal modes, despite the fact that $x = L$ corresponds to the free end of the beam.

Table 7 shows the necessary $(\omega_{osc})_j$ in order to impose nodes at $x_1 = 0{\cdot}33L$, $x_2 = 0{\cdot}49L$ and $x_3 = 0{\cdot}57L$ for a fixed–free beam. The first five natural frequencies of the combined system for each $(\omega_{osc})_j$ are also listed, for $m_1 = m_2 = m_3 = \rho L$. Figure 3(b) shows the fourth and fifth normal modes of the combined assembly, whose attached oscillators have natural frequencies of $(\omega_{osc})_1$ and $(\omega_{osc})_2$ respectively. With the chosen set of node locations, the fourth normal mode has practically zero amplitude from $x = 0$ to $0{\cdot}57L$ along the beam, while the fifth normal mode only undergoes small motion within the same region. Thus,

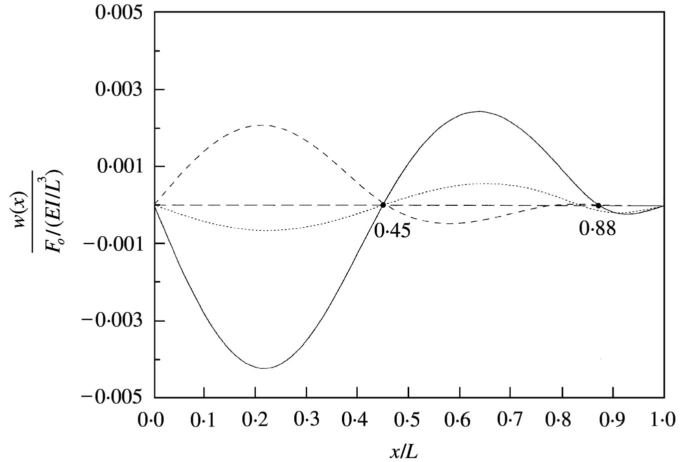

Figure 4. The steady state lateral displacement fields of a simply supported beam subjected to a concentrated harmonic force at $x = 0{\cdot}78L$, for (non-dimensionalized) excitation frequencies of $\omega = 50$, (·····), 55 (——) and 60 (– – –). The system parameters are identical to those of Table 2.

with an appropriate set of node locations, it is possible to force a region of the beam to have practically zero amplitudes without using any rigid supports.

An approach has been proposed to solve the inverse problem of imposing nodes for any normal mode at multiple locations of an arbitrarily supported linear elastic structure. Once the appropriate sprung mass parameters are selected, nodes can be induced for any normal mode at any specified locations along the structure. Under certain conditions, the previous results can be extended to impose regions of small displacements anywhere along the structure when the elastica is being subjected to a harmonic excitation. In general, all the normal modes are excited when a system is forced. However, if the natural frequencies of the system are well separated, and if the excitation frequency centers around the jth natural frequency of the structure, then it is possible to excite only the jth normal mode and leave all the other normal modes totally unaffected. Physically, this implies that when the natural frequencies are far apart and that when the excitation frequency is in the vicinity of a given natural frequency, the structure under forced harmonic response will experience negligible vibration in the regions near the nodes of that particular normal mode.

Consider a simply supported beam, whose system parameters are identical to those of Table 2, subjected to a localized harmonic excitation at $x = 0{\cdot}78L$ with a forcing magnitude of F_o. Figure 4 shows the steady state lateral displacement of the beam as a function of x/L, for excitation frequencies near the third (non-dimensionalized) natural frequency, $\omega_3 = 56{\cdot}3538$, of the system. Note that as expected, the response shape of the beam for each excitation frequency resembles the third normal mode of the beam (see Figure 2(a)), and the displacement fields are all relatively small in the regions near the nodes of the third normal mode, i.e., at $x = 0{\cdot}45L$ and $0{\cdot}88L$ (see Figure 2(a)). Thus, while the focus of this paper has been on imposing nodes to the free vibration of a linear elastic structure, under certain conditions, the approach can be extended to impose regions of small responses during forced vibration, making the proposed scheme a viable means of quenching excess vibrations.

CONCLUSIONS

Elastically mounted masses can be used as a passive means of imposing multiple nodes for any normal mode of a linear elastic structure. When the parameters of the sprung masses are carefully chosen, nodes are induced at the attachment locations, which can be specified anywhere along the combined structure. In addition, if the node locations are properly selected, a region of nearly zero amplitudes may be imposed for a particular normal mode without using any rigid supports. A detailed procedure to assist in the selection of the attached spring–mass systems was outlined, and numerical experiments were performed to validate the utility of the proposed scheme of imposing multiple nodes.

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